

The PLANAR MINLA is different from the MINLA

Alexander Setzer

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Abstract

In various research papers, such as [2], one will find the claim that the MINLA is optimally solvable on outerplanar graphs, with a reference to [1]. However, the problem solved in that publication, which we refer to as the PLANAR MINLA, is different from the MINLA, as we show in this article.

In contrast to the minimum linear arrangement problem (MINLA), the planar minimum linear arrangement problem (PLANAR MINLA) poses an additional restriction on the arrangements: It must be possible to draw all edges of the input graph G such that they “run above” the nodes and do not intersect. More formally:

Definition 1 (Crossing edges). *Let $G = (V, E)$ be a graph and let π be a linear arrangement of G . Two distinct edges $\{u, v\}, \{x, y\} \in E$ cross if: $\pi(u) < \pi(x) < \pi(v) < \pi(y)$.*

Definition 2 (Minimum planar arrangement). *A minimum planar arrangement of an input graph $G = (V, E)$ is a mapping $\pi : V \rightarrow \{1, \dots, n\}$ such that no two edges of G cross in π .*

We prove that optimal solutions of the PLANAR MINLA are different from optimal solutions of the MINLA by presenting a counterexample. That is, we give an example graph whose corresponding minimum linear arrangement yields a smaller cost than all possible minimum planar arrangements.

The input graph $G = (V, E)$ we use is given by Figure 1.

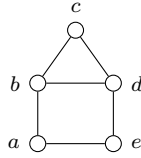


Figure 1: Input graph used to prove the counterexample.

We claim that the arrangement π_1 given by Figure 2 yields a lower cost than any minimum planar arrangement.

Theorem 3. *Any minimum planar arrangement of G has a cost strictly larger than 9.*

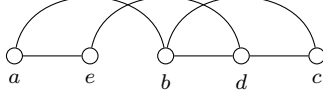


Figure 2: Arrangement π_1 of G . The cost of π_1 is 9.

In order to prove this, we determine all minimum planar arrangements of G (which are exactly five, plus their symmetric counterparts, as we will see). For this, we need the following terminology, which is taken from [1]:

Definition 4 (Dominating edge). *Let $G = (V, E)$ be a graph and let π_p be a minimum planar arrangement of G . An edge $\{x, y\}$ dominates an edge $\{u, v\}$, if $\{x, y\} \neq \{u, v\}$ and $\pi_p(u) \leq \pi_p(x) < \pi_p(y) \leq \pi_p(v)$.*

Provided with this definition, we are ready to prove Theorem 3:

Proof of Theorem 3: The idea of this proof is to identify all possible minimum planar arrangements and to show that the cost of each such arrangement is greater than 9. Instead of testing all permutations of the nodes in G , we find conditions of minimum planar arrangements for G and then only consider graphs that fulfill these conditions.

Let π_p be a minimum planar arrangement of G . Moreover, let E_C be the set of edges on the cycle (a, b, c, d, e, a) in G . We first show:

1. For each edge $\{u, v\} \in E_C$, one of the following is true:
 - (a) $\{u, v\}$ dominates all other edges in π_p
 - (b) u and v are neighbors in π_p
2. There is exactly one edge $\{u, v\} \in E_C$ that dominates all other edges.

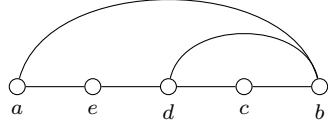
For the first claim, assume that for an arbitrary edge $\{u, v\} \in E_C$, none of the two cases is true, i.e., neither does $\{u, v\}$ dominate all other edges, nor are u and v neighbors in π_p . This implies there are exactly one or two nodes between u and v . However, observe that for each of the edges $\{u, v\} \in E_C$, the subgraph induced by $V \setminus \{u, v\}$ contains a path of length three. Independent of which single or two nodes we place between u and v , this path crosses $\{u, v\}$. Therefore, this is not possible without violating the constraints of a minimum planar arrangement and the first claim is proven.

For the second claim, assume for contradiction that the claim is not true, i.e. there is no edge $\{u, v\} \in E_C$ that dominates all other edges. This implies, by the first claim, for each edge $\{x, y\} \in E_C$ that x and y are neighbors in π_p . However, since the edges in E_C form a cycle, this is not possible for all edges. This completes the proof of the second claim.

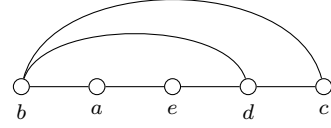
Now, let $\{u, v\} \in E_C$ be the edge that dominates all other edges in π_p . The first claim implies that for the edges $\{x, y\} \in E_C \setminus \{u, v\}$, x and y must be neighbors in π_p . Since the edges in $E_C \setminus \{u, v\}$ form a path from u to v , this uniquely defines the order of the other nodes in π_p .

Provided with this, we can derive all possible minimum planar arrangements by selecting an edge $\{u, v\} \in E_C$, putting u and v at the positions 1 and 5, and placing the other nodes such that for $\{x, y\} \in E_C \setminus \{u, v\}$, x and y are neighbors

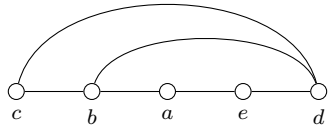
in the arrangement. This yields five possible minimum planar arrangements (except for symmetry):



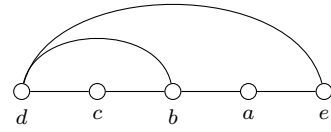
(a) Arrangement with a and b at the outmost positions. Its cost is 10.



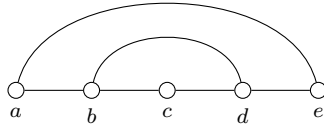
(b) Arrangement with b and c at the outmost positions. Its cost is 11.



(c) Arrangement with c and d at the outmost positions. Its cost is 11.



(d) Arrangement with d and e at the outmost positions. Its cost is 10.



(e) Arrangement with a and e at the outmost positions. Its cost is 10.

Figure 3: The possible minimum planar arrangements of G .

All these arrangements have a cost of more than 9. □

References

- [1] Greg N. Frederickson and Susanne E. Hambrusch. Planar linear arrangements of outerplanar graphs. *Circuits and Systems, IEEE Transactions on*, 35(3):323–333, 1988.
- [2] Jordi Petit. Addenda to the survey of layout problems. *Bulletin of the EATCS*, (105):177–201, 2011.